On the analytic reduction of singularly perturbed differential equations

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March 24, 2015

Outline

1. Introduction and results
2. Gevrey theory of CAsEs
3. Proof of the main result

Consider the differential equation

$$\varepsilon^2 \frac{d^2 y}{dx^2} - Q(x)y = 0,$$

where

- $\varepsilon > 0$, $\varepsilon \to 0$,
- $x \in [a, b]$,
- $Q : [a, b] \to \mathbb{R}$ of class C^1.

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Example

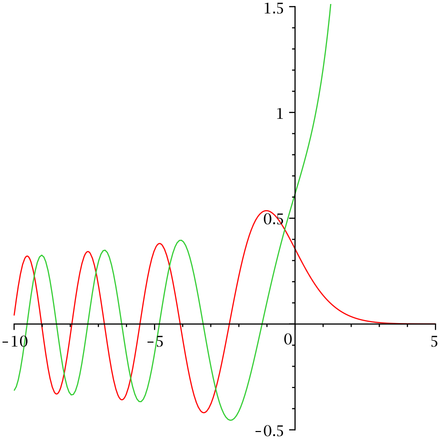
The Schrödinger equation (1925) :

$$\frac{d^2 y}{dx^2} - \frac{2m}{\hbar^2}(V(x) - E) y = 0.$$

Here $\hbar$ plays the role of ε and $Q(x) = 2m(V(x) - E)$.

Turning point

The zeros of $Q(x)$ separate regions with oscillating behavior from regions with exponential behavior.



Mathematical background

Consider the differential equation

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y, \qquad (1)$$

where

- x is a complex variable,
- ε is a small complex parameter,
- A is a 2×2 matrix of holomorphic and bounded functions on $D(0, r_0) \times D(0, \varepsilon_0)$.

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The case «$A(0,0)$ admits two distinct eigenvalues» is well known.

Otherwise the point $x = 0$ is a turning point for equation (1).

Mathematical background

In this talk, we consider differential equations

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y,$$

where

- $\operatorname{tr} A(x,\varepsilon) \equiv 0$,
- $A_0(x) := A(x,0) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix}$, with $\mu, \nu \in \mathbb{N}$ and $\mu + \nu \neq 0$.

Theorems of simplification

Hanson & Russell (1967)

Theorem. *There exists a formal power series*

$$\hat{T}(x,\varepsilon) = \sum_{n\geq 0} T_n(x)\varepsilon^n$$

such that $\det T_0(x) \equiv 1$ *and*

$$\varepsilon\frac{dy}{dx} = A(x,\varepsilon)y \quad \underset{y=\hat{T}(x,\varepsilon)z}{\sim} \quad \varepsilon\frac{dz}{dx} = \hat{B}(x,\varepsilon)z$$

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where

$$\hat{B}(x,\varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} \hat{b}_{11}(x,\varepsilon) & \hat{b}_{12}(x,\varepsilon) \\ \hat{b}_{21}(x,\varepsilon) & \hat{b}_{22}(x,\varepsilon) \end{pmatrix}$$

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$$\deg_x \hat{b}_{11} < \mu, \quad \deg_x \hat{b}_{12} < \mu, \quad \deg_x \hat{b}_{21} < \mu+\nu \quad \textit{and} \quad \deg_x \hat{b}_{22} < \mu.$$

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- $$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y \quad \underset{y = T(x,\varepsilon) z}{\sim} \quad \varepsilon \frac{dz}{dx} = B(x, \varepsilon) z$$

  where

  $$B(x, \varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} b_{11}(x, \varepsilon) & b_{12}(x, \varepsilon) \\ b_{21}(x, \varepsilon) & -b_{11}(x, \varepsilon) \end{pmatrix}$$

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  and the b_{ij} are polynomials in x :

  $$\deg_x b_{11} < \mu, \quad \deg_x b_{12} < \mu \quad \text{and} \quad \deg_x b_{21} < \mu + \nu.$$

Known results

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$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y \quad \text{and} \quad A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix}.$$

The case $\mu = 0$ is well known :

- Wasow treated the case $A_0(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$ in 1965,
- Lee treated the case $A_0(x) = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ in 1969,
- Sibuya treated the case $A_0(x) = \begin{pmatrix} 0 & 1 \\ x^{\nu} & 0 \end{pmatrix}$, $\nu \in \mathbb{N}^{\star}$, in 1974.

Gevrey theory of composite asymptotic expansions

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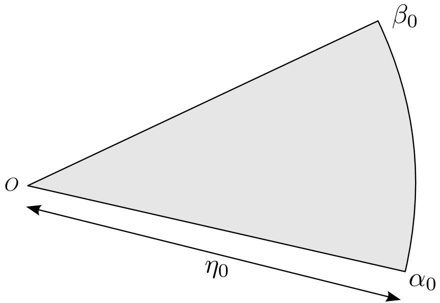
- $S = \{\eta \in \mathbb{C},\ 0 < |\eta| < \eta_0 \text{ and } \alpha_0 < \arg \eta < \beta_0\}$,



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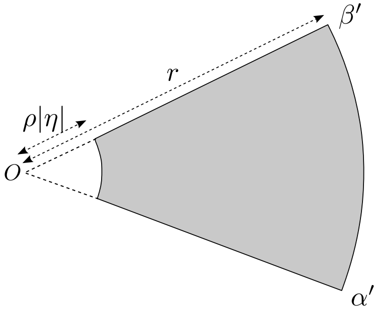

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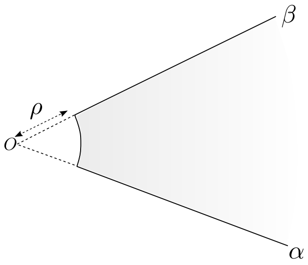
- $V = \{\mathbf{X} \in \mathbb{C},\ \rho < |\mathbf{X}| \text{ and } \alpha < \arg \mathbf{X} < \beta\}$.



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Remark.

$$\text{If } \eta \in S \text{ and } x \in V(\eta), \text{ then } \mathbf{X} = \frac{x}{\eta} \in V.$$

Formal composite series

Definition

A *formal composite series* associated to V and $D(0,r)$ is a series of this form

$$\hat{y}(x,\eta) = \sum_{n\geq 0} \left(a_n(x) + g_n(\tfrac{x}{\eta})\right) \eta^n$$

such that $\forall n \in \mathbb{N}$,
a_n is holomorphic and bounded on $D(0,r)$,
g_n is holomorphic and bounded on V and

$$g_n(\mathbf{X}) \sim \sum_{m>0} g_{nm}\mathbf{X}^{-m}, \text{ as } V \ni \mathbf{X} \to \infty.$$

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The series $\sum_{n\geq 0} a_n(x)\eta^n$ is called the slow part of $\hat{y}(x,\eta)$.
The series $\sum_{n\geq 0} g_n(\frac{x}{\eta})\eta^n$ is called the fast part of $\hat{y}(x,\eta)$.

Outer and inner expansions

$$\hat{y}(x,\eta) = \sum_{n\geq 0} \left(a_n(x) + g_n(\tfrac{x}{\eta}) \right) \eta^n$$

How can we determine the $a_n(x)$ and the $g_n(X)$?

Outer and inner expansions

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How can we determine the $a_n(x)$ and the $g_n(X)$?

For fixed non-zero x, one computes the outer expansion

$$y(x,\eta) \sim \sum_{n\geq 0} c_n(x)\eta^n,$$

then one eliminates the terms with negative powers of x to obtain the slow parts $a_n(x)$:

$$c_n(x) \rightsquigarrow a_n(x).$$

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How can we determine the $a_n(x)$ and the $g_n(X)$?

Analogously, one computes the inner expansion

$$y(\eta X, \eta) \sim \sum_{n\geq 0} h_n(X) \eta^n,$$

then one eliminates the terms with non-negative powers of X to obtain the fast parts $g_n(X)$:

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Let $y(x,\eta)$ be holomorphic and bounded for $\eta \in S$ and for $x \in V(\eta)$, and let

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Definition

We say that y admits $\hat{y}$ as *composite asymptotic expansion* (CAsE), as $\eta \to 0$ in S and $x \in V(\eta)$, if $\forall N \in \mathbb{N}$, $\exists K_N > 0$,

$$\left| y(x, \eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n(\tfrac{x}{\eta})\right) \eta^n \right| \leq K_N |\eta|^N,$$

for all $\eta \in S$ and all $x \in V(\eta)$.

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Definition

We say that y admits $\hat{y}$ as *CAsE of Gevrey order* $\frac{1}{p}$, as $\eta \to 0$ in S and $x \in V(\eta)$, if $\exists C, L > 0$, $\forall N \in \mathbb{N}$,

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n(\tfrac{x}{\eta})\right) \eta^n \right| \leq CL^N \Gamma(\tfrac{N}{p} + 1) |\eta|^N,$$

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for all $\eta \in S$ and all $x \in V(\eta)$.

Notation: $y(x,\eta) \sim_{\frac{1}{p}} \hat{y}(x,\eta)$, as $\eta \to 0$ in S and $x \in V(\eta)$.

Proof of the main result

Assume that ν is even : $\nu = 2\gamma$.

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We consider a differential equation

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon) y,$$

where

$$A(x,0) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix}.$$

Fundamental matrix solution

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Proposition. *The differential equation* $\varepsilon\frac{dy}{dx} = A(x,\varepsilon)y$ *has a fundamental matrix solution of the form*

$$Y(x,\eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} Q(x,\eta)e^{\Lambda(x,\eta)}$$

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where
η *is a certain root of* ε, $\varepsilon = \eta^p$,
Q *admits a CAsE of Gevrey order* $\frac{1}{p}$,
Λ *is a diagonal matrix :*

$$\Lambda(x,\eta) = \begin{pmatrix} -\frac{1}{p}\frac{x^p}{\eta^p} + R_1(\varepsilon)\log x & 0 \\ 0 & \frac{1}{p}\frac{x^p}{\eta^p} + R_2(\varepsilon)\log x \end{pmatrix}.$$

Slow-fast factorization

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Theorem. *There exist* $L(x,\varepsilon)$ *holomorphic and bounded on* $D(0,r)\times\tilde{S}$ *and* $R(x,\eta)$ *holomorphic and bounded for* $\eta\in S$*,* $x\in V(\eta)$*, such that :*

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$$Q(x, \eta) = L(x, \varepsilon) \cdot R(x, \eta),$$

$$L(x, \varepsilon) \sim_1 \sum_{n \geq 0} A_n(x) \varepsilon^n \quad \text{as } \varepsilon \to 0 \text{ in } \tilde{S} \text{ and } |x| < r,$$

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$$Q(x,\eta)=L(x,\varepsilon)\cdot R(x,\eta),$$

$L(x,\varepsilon)\sim_1 \sum_{n\geq 0} A_n(x)\varepsilon^n$ *as* $\varepsilon\to 0$ *in* $\tilde{S}$ *and* $|x|<r$*,*

and

$R(x,\eta)\sim_{\frac{1}{p}} \sum_{n\geq 0} G_n(\frac{x}{\eta})\eta^n$ *as* $\eta\to 0$ *in* S *and* $x\in V(\eta)$*.*

Slow-fast factorization

Preparation of Y

Slow-fast factorization

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As $Q = L \cdot R$, we have

$$
\begin{aligned}
Y(x,\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} Q(x,\eta) e^{\Lambda(x,\varepsilon)}, \\
&= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} L(x,\varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & x^{-\gamma} \end{pmatrix}}_{P(x,\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} R(x,\eta) e^{\Lambda(x,\varepsilon)}.
\end{aligned}
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Slow-fast factorization

Preparation of Y

The matrix $Y(x,\eta)$ can be written

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where

P is a slow matrix, i.e.

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R is a fast matrix, i.e.

$$R(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0} G_n\left(\frac{x}{\eta}\right)\eta^n \quad \text{as } S \ni \eta \to 0,\ x \in V(\eta),$$

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Analytic simplification

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Proposition. *The change of variables* $y = P(x,\varepsilon)z$ *reduces the differential equation* $\varepsilon\frac{dy}{dx} = A(x,\varepsilon)y$ *to*

$$\varepsilon\frac{dz}{dx} = B(x,\varepsilon)z,$$

where $B(x,\varepsilon) \sim_1 \hat{B}(x,\varepsilon)$*,*

$$\hat{B}(x,\varepsilon) = \begin{pmatrix} \hat{b}_{11}(x,\varepsilon) & \hat{b}_{12}(x,\varepsilon) \\ \hat{b}_{21}(x,\varepsilon) & -\hat{b}_{11}(x,\varepsilon) \end{pmatrix},$$

and the $\hat{b}_{ij}$ *are polynomials in* x*.*

Analytic simplification

Proof of the proposition

Proof. On the one hand,

$$B = P^{-1}AP - \varepsilon P^{-1}P'$$

and

$$B(x,\varepsilon) \sim_1 \hat{B}(x,\varepsilon).$$

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On the other hand, $Z(x,\eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} R(x,\eta)e^{\Lambda(x,\eta)}$ is a fundamental matrix solution of equation $\varepsilon\frac{dz}{dx} = B(x,\varepsilon)z$ and

$$B(x,\varepsilon) = \varepsilon Z'(x,\eta)Z(x,\eta)^{-1}.$$

We deduce a bound for the degree of each entry of $\hat{B}(x,\varepsilon)$.

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Thank you for your attention !

Condition $(\mathcal{C})$

We consider a differential equation

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where

$$A(x,\varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{a}(x,\varepsilon) & \mathbf{b}(x,\varepsilon) \\ \mathbf{c}(x,\varepsilon) & -\mathbf{a}(x,\varepsilon) \end{pmatrix}.$$

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Condition $(\mathcal{C})$:

1. ν is even and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-2)})$,
2. ν is odd and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-1)})$.