Uniform simplification and summability

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Outline

Consider the differential equation

$$\varepsilon^2 \frac{d^2 y}{dx^2} - Q(x)y = 0,$$

where

- $\varepsilon > 0$, $\varepsilon \to 0$,
- $x \in [a, b]$,
- $Q : [a, b] \to \mathbb{R}$ of class C^1.

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Example

The Schrödinger equation (1925) :

$$\frac{d^2 y}{dx^2} - \frac{2m}{\hbar^2}(V(x) - E)y = 0.$$

Here $\hbar$ plays the role of ε and $Q(x) = 2m(V(x) - E)$.

Liouville-Green (1837)

$$\varepsilon^2 \frac{d^2 y}{dx^2} - Q(x)y = 0 \tag{1}$$

If $Q(x) > 0$,

$$\phi^{\pm}(x, \varepsilon) = Q(x)^{-\frac{1}{4}} \exp\left(\pm \frac{1}{\varepsilon} \int^x \sqrt{Q(\xi)} d\xi \right). \tag{2}$$

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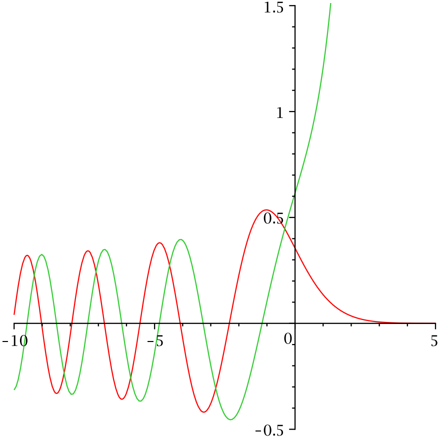
At a zero of Q, the functions (2) and (3) are not approximations of the solutions anymore.

Turning point

The zeros of Q separate regions with oscillating behavior from regions with exponential behavior.



Turning point

The zeros of $Q(x)$ separate regions with oscillating behavior from regions with exponential behavior.

Definition

The zeros of Q are called *turning points*.

Mathematical background

Consider the differential equation

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y, \tag{1}$$

where

- x is a complex variable,
- ε is a small complex parameter,
- A is a 2×2 matrix of holomorphic and bounded functions on $D(0, r_0) \times D(0, \varepsilon_0)$.

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Otherwise the point $x = 0$ is a turning point for equation (1).

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- $\operatorname{tr} A(x,\varepsilon) \equiv 0$,
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- $\det A_0(x) \not\equiv 0$.

In this case $A_0(x)$ admits two distinct eigenvalues when $x \neq 0$, which are equal at $x = 0$.

Assumptions

We can reduce the study to differential equations of this form

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y,$$

where

- $\operatorname{tr} A(x, \varepsilon) \equiv 0$,
- $A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix}$, with $\mu, \nu \in \mathbb{N}$ and $\mu + \nu \neq 0$.

Condition ($\mathcal{C}$)

We consider a differential equation

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where

$$A(x,\varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{a}(x,\varepsilon) & \mathbf{b}(x,\varepsilon) \\ \mathbf{c}(x,\varepsilon) & -\mathbf{a}(x,\varepsilon) \end{pmatrix}.$$

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Condition $(\mathcal{C})$:

1. ν is even and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-2)})$,
2. ν is odd and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-1)})$.

Theorems of simplification

Hanson & Russell (1967)

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Theorem. *There exists a formal power series*

$$\hat{T}(x,\varepsilon) = \sum_{n\geq 0} T_n(x)\varepsilon^n$$

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where

$$\hat{B}(x,\varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} \hat{b}_{11}(x,\varepsilon) & \hat{b}_{12}(x,\varepsilon) \\ \hat{b}_{21}(x,\varepsilon) & \hat{b}_{22}(x,\varepsilon) \end{pmatrix}$$

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$$\deg_x \hat{b}_{11} < \mu, \quad \deg_x \hat{b}_{12} < \mu, \quad \deg_x \hat{b}_{21} < \mu+\nu \quad \textit{and} \quad \deg_x \hat{b}_{22} < \mu.$$

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- $\det T_0(x) \equiv 1$,
- $$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y \quad \underset{y = T(x,\varepsilon) z}{\sim} \quad \varepsilon \frac{dz}{dx} = B(x, \varepsilon) z$$

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  $$\deg_x b_{11} < \mu, \quad \deg_x b_{12} < \mu \quad \text{and} \quad \deg_x b_{21} < \mu + \nu.$$

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- Sibuya treated the case $A_0(x) = \begin{pmatrix} 0 & 1 \\ x^{\nu} & 0 \end{pmatrix}$, $\nu \in \mathbb{N}^{\star}$, in 1974.

Gevrey theory of composite asymptotic expansions

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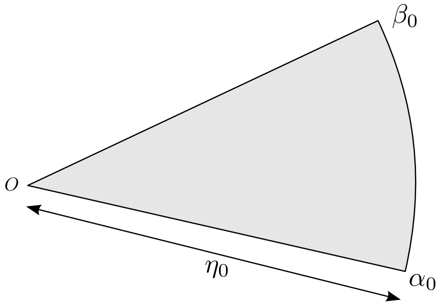
Let

- $S = \{\eta \in \mathbb{C},\ 0 < |\eta| < \eta_0 \text{ and } \alpha_0 < \arg \eta < \beta_0\}$,



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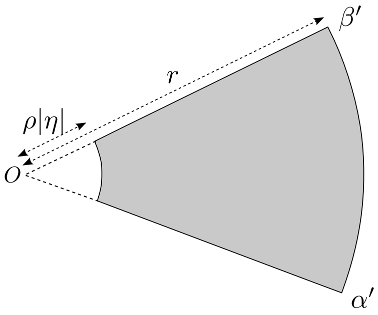

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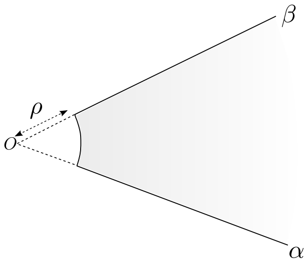

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- $V = \{\mathbf{X} \in \mathbb{C},\ \rho < |\mathbf{X}| \text{ and } \alpha < \arg \mathbf{X} < \beta\}$.

We call $(\mathcal{P})$ the following property :

$$\text{If } \eta \in S \text{ and } x \in V(\eta), \text{ then } \mathbf{X} = \frac{x}{\eta} \in V.$$

Formal composite series

Definition

A *formal composite series* associated to V and $D(0,r)$ is a series of this form

$$\hat{y}(x,\eta) = \sum_{n\geq 0} \left(a_n(x) + g_n\left(\tfrac{x}{\eta}\right) \right) \eta^n$$

such that $\forall n \in \mathbb{N}$,
a_n is holomorphic and bounded on $D(0,r)$,
g_n is holomorphic and bounded on V and

$$g_n(\mathbf{X}) \sim \sum_{m>0} g_{nm} \mathbf{X}^{-m}, \text{ as } V \ni \mathbf{X} \to \infty.$$

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The series $\sum_{n\geq 0} a_n(x)\eta^n$ is called the slow part of $\hat{y}(x,\eta)$.
The series $\sum_{n\geq 0} g_n(\frac{x}{\eta})\eta^n$ is called the fast part of $\hat{y}(x,\eta)$.

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Let $y(x,\eta)$ be holomorphic and bounded for $\eta \in S$ and for $x \in V(\eta)$, and let

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Definition

We say that y admits $\hat{y}$ as *composite asymptotic expansion* (CAsE), as $\eta \to 0$ in S and $x \in V(\eta)$, if $\forall N \in \mathbb{N}$, $\exists K_N > 0$,

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n(\tfrac{x}{\eta})\right) \eta^n \right| \leq K_N |\eta|^N,$$

for all $\eta \in S$ and all $x \in V(\eta)$.

Gevrey CAsE

Definition

We say that y admits $\hat{y}$ as *CAsE of Gevrey order* $\frac{1}{p}$, as $\eta \to 0$ in S and $x \in V(\eta)$, if $\exists C, L > 0$, $\forall N \in \mathbb{N}$,

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n(\tfrac{x}{\eta})\right) \eta^n \right| \leq CL^N \Gamma(\tfrac{N}{p} + 1)|\eta|^N,$$

for all $\eta \in S$ and all $x \in V(\eta)$ and

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Notation: $y(x,\eta) \sim_{\frac{1}{p}} \hat{y}(x,\eta)$, as $\eta \to 0$ in S and $x \in V(\eta)$.

Consistent good covering

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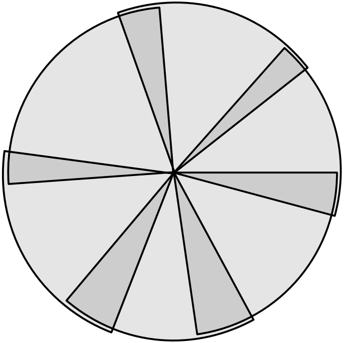

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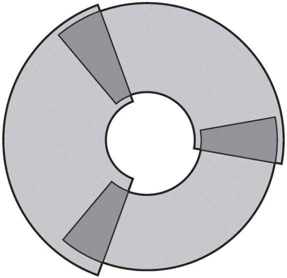

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- for all $\eta \in S_\ell$,
  $(V_\ell^j(\eta))_j$ is a consistent good covering of
  $\{x \in \mathbb{C},\ \rho|\eta| < |x| < r\}$,
- if $\eta \in S_\ell$ and $x \in V_\ell^j(\eta)$, then $\frac{x}{\eta} \in V^j$.

Theorem of Fruchard-Schäfke

A theorem of Ramis-Sibuya type

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A theorem of Ramis-Sibuya type

Let $S_\ell, V^j, V^j_\ell(\eta), \ell = 1, \ldots, L, j = 1, \ldots, J$, be a consistent good covering.

Theorem of Fruchard-Schäfke
A theorem of Ramis-Sibuya type

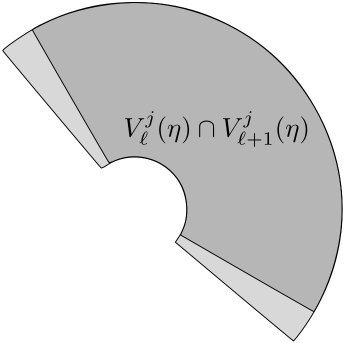
Let S_ℓ, V^j, $V_\ell^j(\eta)$, $\ell = 1, \dots, L$, $j = 1, \dots, J$, be a consistent good covering.
Let $\left(y_\ell^j(x,\eta)\right)_{j,\ell}$ be a collection of holomorphic and bounded functions defined for $\eta \in S_\ell$ and $x \in V_\ell^j(\eta)$.

Theorem of Fruchard-Schäfke

A theorem of Ramis-Sibuya type

If

$$\left|\left(y^j_{\ell+1} - y^j_\ell\right)(x,\eta)\right| = \mathcal{O}\left(e^{-\frac{A}{|\eta|^p}}\right)$$



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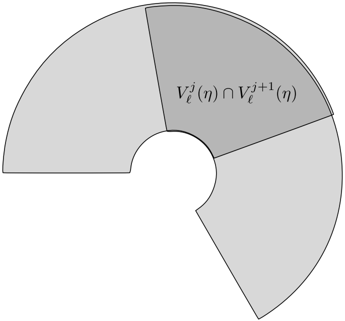

Theorem of Fruchard-Schäfke

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If

$$\left|\left(y^j_{\ell+1} - y^j_\ell\right)(x,\eta)\right| = \mathcal{O}\left(\mathrm{e}^{-\frac{A}{|\eta|^p}}\right)$$

and

$$\left|\left(y^{j+1}_\ell - y^j_\ell\right)(x,\eta)\right| = \mathcal{O}\left(\mathrm{e}^{-B\left|\frac{x}{\eta}\right|^p}\right),$$

then

$$y^j_\ell(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0}\left(a_n(x) + g^j_n(\tfrac{x}{\eta})\right)\eta^n,$$

$$g^j_n(\mathbf{X}) \sim_{\frac{1}{p}} \sum_{m>0} g_{nm}\mathbf{X}^{-m}, \text{ as } V^j \ni \mathbf{X} \to \infty.$$

Proof of the main result

Assume that ν is even : $\nu = 2\gamma$.

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We consider a differential equation

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y,$$

where

$$A(x,\varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{a}(x,\varepsilon) & \mathbf{b}(x,\varepsilon) \\ \mathbf{c}(x,\varepsilon) & -\mathbf{a}(x,\varepsilon) \end{pmatrix}.$$

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In this case, the condition $(\mathcal{C})$ becomes $\mathbf{c}(x,0) = \mathcal{O}(x^{\gamma-1})$.

Steps of the proof

- Fundamental matrix solution

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- Analytic simplification

Fundamental matrix solution

Fundamental matrix solution

Proposition. *The differential equation* $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$ *has a fundamental matrix solution of the form*

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Λ *is a diagonal matrix.*

Fundamental matrix solution

Preparation of equation (1)

$$(1) \quad \varepsilon \frac{dy}{dx} = A(x,\varepsilon) y \quad \text{where } A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix},$$

$$\downarrow \qquad y = T(x) u$$

$$(2) \quad \varepsilon \frac{du}{dx} = B(x,\varepsilon) u \quad \text{where } B_0(x) = \begin{pmatrix} -x^{p-1} & 0 \\ 0 & x^{p-1} \end{pmatrix},$$

$$\downarrow \qquad u = \Phi(x,\eta) v \text{ and } \varepsilon = \eta^p$$

$$(3) \quad \eta^p \frac{dv}{dx} = C(x,\eta) v \quad \text{where } C(x,\eta) = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}.$$

Fundamental matrix solution

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Fundamental matrix solution

Existence of Φ

We now make the second change of variables explicit:

$$u = \Phi v \quad \text{and} \quad \varepsilon = \eta^p.$$

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The matrix Φ is as follows :

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The function ϕ^+, resp. ϕ^-, satisfies a Riccati equation :

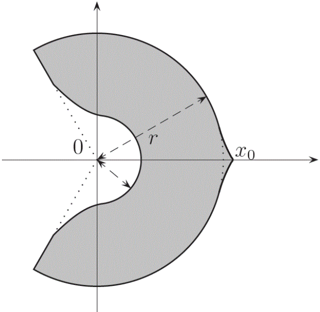
$$\eta^p \frac{d\phi}{dx} = \pm 2x^{p-1}\phi + F^{\pm}(\phi)(x,\eta).$$

Fundamental matrix solution

Existence of ϕ^+

$$\eta^p \frac{d\phi^+}{dx} = 2x^{p-1}\phi^+ + F^+(\phi^+)$$

Let $\mathcal{M}_k$ be the set of holomorphic functions $\phi(x, \eta)$ defined for $\eta \in S$ and $x \in \Omega(\eta)$ such that $|\phi(x, \eta)| \leq k$.



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$$\phi \quad \mapsto \quad \frac{1}{\eta^p} \int_{x_0}^{x} e^{\frac{2}{p}\left(\frac{x^p}{\eta^p} - \frac{\xi^p}{\eta^p}\right)} F^+(\phi(\xi,\eta))\, d\xi.$$

Fundamental matrix solution

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- existence of $(\phi^+)^j_\ell$
- $(\phi^+)^j_\ell(x,\eta) \sim_{\frac{1}{p}} (\hat{\phi}^+)^j(x,\eta)$

Fundamental matrix solution

Summary

(1) $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$

$\downarrow \qquad y = T(x)u$

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Fundamental matrix solution

Summary

(1)	$\varepsilon \frac{dy}{dx} = A(x,\varepsilon) y$		$Y = T \Phi V$
	$\downarrow$	$y = T(x) u$	$\uparrow$
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We deduce the form of a fundamental matrix solution of $\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y$:

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Q admits a CAsE of Gevrey order $\frac{1}{p}$,

$$\Lambda(x,\eta) = \begin{pmatrix} -\frac{1}{p}\frac{x^p}{\eta^p} + R_1(\varepsilon)\log x & 0 \\ 0 & \frac{1}{p}\frac{x^p}{\eta^p} + R_2(\varepsilon)\log x \end{pmatrix}.$$

Slow-fast factorization

Slow-fast factorization

Theorem. *For all* $r \in]0, r_0[$, *there exist* $L(x, \varepsilon)$ *holomorphic and bounded on* $D(0, r) \times \tilde{S}$ *and* $R(x, \eta)$ *holomorphic and bounded for* $\eta \in S$, $x \in V(\eta)$, *such that :*

$$Q(x, \eta) = L(x, \varepsilon) \cdot R(x, \eta),$$

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and

$$R(x,\eta) \sim_{\frac{1}{p}} \textstyle\sum_{n\geq 0} G_n(\frac{x}{\eta})\eta^n \quad \text{as } \eta \to 0 \text{ in } S \text{ and } x \in V(\eta),$$

$$G_n(\mathbf{X}) \sim_{\frac{1}{p}} \sum_{m\geq 0} G_{nm}\mathbf{X}^{-m} \quad \text{as } \mathbf{X} \to \infty \text{ in } V.$$

Slow-fast factorization

Preparation of Y

Slow-fast factorization

Preparation of Y

As $Q = L \cdot R$, we have

$$
\begin{aligned}
Y(x,\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} Q(x,\eta) e^{\Lambda(x,\varepsilon)}, \\
&= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} L(x,\varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & x^{-\gamma} \end{pmatrix}}_{P(x,\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} R(x,\eta) e^{\Lambda(x,\varepsilon)}.
\end{aligned}
$$

Slow-fast factorization

Preparation of Y

The matrix $Y(x,\eta)$ can be written

$$Y(x,\eta) = P(x,\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} R(x,\eta)\mathrm{e}^{\Lambda(x,\varepsilon)},$$

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P is a slow matrix, i.e.

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R is a fast matrix, i.e.

$$R(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0} G_n(\tfrac{x}{\eta})\eta^n \quad \text{as } S \ni \eta \to 0,\ x \in V(\eta),$$

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Analytic simplification

Analytic simplification

Proposition. *The change of variables $y = P(x, \varepsilon)w$ reduces the differential equation $\varepsilon \frac{dy}{dx} = A(x, \varepsilon)y$ to*

$$\varepsilon \frac{dw}{dx} = D(x, \varepsilon)w,$$

where $D(x, \varepsilon) \sim_1 \hat{D}(x, \varepsilon)$,

$$\hat{D}(x, \varepsilon) = \begin{pmatrix} \hat{d}_{11}(x, \varepsilon) & \hat{d}_{12}(x, \varepsilon) \\ \hat{d}_{21}(x, \varepsilon) & -\hat{d}_{11}(x, \varepsilon) \end{pmatrix},$$

and the $\hat{d}_{ij}$ are polynomials in x such that

$$\deg_x \hat{d}_{11} \leq \mu + \gamma, \quad \deg_x \hat{d}_{12} = \mu \quad \textit{and} \quad \deg_x \hat{d}_{21} = \mu + 2\gamma.$$

Analytic simplification

Proof of the proposition

Proof. On the one hand,

$$D = P^{-1}AP - \varepsilon P^{-1}P'$$

and

$$D(x,\varepsilon) \sim_1 \hat{D}(x,\varepsilon),$$

as $\varepsilon \to 0$ in $\tilde{S}$ and $|x| < r$.

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On the other hand, $W(x,\eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} R(x,\eta)\mathrm{e}^{\Lambda(x,\eta)}$ is a fundamental matrix solution of equation $\varepsilon\frac{dw}{dx} = D(x,\varepsilon)w$ and

$$D(x,\varepsilon) = \varepsilon W'(x,\eta)W(x,\eta)^{-1}.$$

We deduce a bound for the degree of each entry of $\hat{D}(x,\varepsilon)$.

Analytic simplification
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We deduce a bound for the degree of each entry of $\hat{D}(x,\varepsilon)$.

Analytic simplification

Let $\tilde{D} = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \tilde{d}_{22} & -\tilde{d}_{11} \end{pmatrix}$ be a matrix of polynomials in x such that

$$\tilde{D}(x,\varepsilon) \sim_1 \hat{D}(x,\varepsilon) \quad \text{as } \varepsilon \to 0 \text{ in } \tilde{S}$$

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Proposition. *For all $r \in]0, r_0[$, there exists $\tilde{P}(x,\varepsilon)$, holomorphic and bounded on $D(0,r) \times \tilde{S}$, admitting an asymptotic expansion of Gevrey order 1, such that* $\det P_0(x) \equiv 1$ *and the change of variables $y = \tilde{P}(x,\varepsilon)w$ reduces the differential equation $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$ to*

$$\varepsilon \frac{dw}{dx} = \tilde{D}(x,\varepsilon)w.$$

The main result (even case)

Theorem. *If* $(\mathcal{C})$ *is satisfied, then* $\forall r \in]0, r_0[\ ,\ \forall S,\ \exists T(x,\varepsilon)$ *holomorphic and bounded on* $D(0,r) \times S$ *such that :*

- $T(x,\varepsilon) \sim_1 \hat{T}(x,\varepsilon)$ *as* $\varepsilon \to 0$ *in* S *and* $|x| < r$,
- $\det T_0(x) \equiv 1$,
- $$\varepsilon \frac{dy}{dx} = A(x,\varepsilon) y \underset{y=T(x,\varepsilon)z}{\sim} \varepsilon \frac{dz}{dx} = B(x,\varepsilon) z$$

  where

  $$B(x,\varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} b_{11}(x,\varepsilon) & b_{12}(x,\varepsilon) \\ b_{21}(x,\varepsilon) & -b_{11}(x,\varepsilon) \end{pmatrix},$$

  and the b_{ij} *are polynomials in* x *such that*

  $$\deg_x b_{11} < \mu,\ \deg_x b_{12} < \mu \text{ and } \deg_x b_{21} < \mu + 2\gamma.$$

Summability

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Can we obtain a 1-summable simplification in the direction $\arg \varepsilon = 0$?

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Definition

Let $\hat{f}(\varepsilon) = \sum_{n \geq 0} f_n \varepsilon^n$ be a Gevrey-1 formal series.
We say that $\hat{f}(\varepsilon)$ is 1-summable in the direction $\arg \varepsilon = 0$ if there exist $\delta, \varepsilon_0 > 0$ and a holomorphic function $f(\varepsilon)$ on the sector $S = \left\{ \varepsilon \in \mathbb{C},\ 0 < |\varepsilon| < \varepsilon_0 \text{ and } |\arg \varepsilon| < \frac{\pi}{2} + \delta \right\}$ such that $f \sim_1 \hat{f}$.

Summability

The differential equation $\varepsilon\frac{dy}{dx} = A(x,\varepsilon)y$, where

$$A(x,\varepsilon) = \begin{pmatrix} \varepsilon\mathbf{a}(x,\varepsilon) & x^{\mu} + \varepsilon\mathbf{b}(x,\varepsilon) \\ x^{\mu+2\gamma} + \varepsilon\mathbf{c}(x,\varepsilon) & -\varepsilon\mathbf{a}(x,\varepsilon) \end{pmatrix},$$

is formally equivalent to $\varepsilon\frac{dz}{dx} = \hat{B}(x,\varepsilon)z$ via the formal change of variables $y = \hat{T}(x,\varepsilon)z$.

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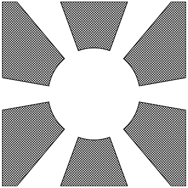
Assumption A1 : $A(x,\varepsilon)$ is analytic on $\mathcal{D} \times D(0,\varepsilon_0)$.



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$$A(x,\varepsilon) = \begin{pmatrix} \varepsilon \mathbf{a}(x,\varepsilon) & x^{\mu} + \varepsilon \mathbf{b}(x,\varepsilon) \\ x^{\mu+2\gamma} + \varepsilon \mathbf{c}(x,\varepsilon) & -\varepsilon \mathbf{a}(x,\varepsilon) \end{pmatrix},$$

is formally equivalent to $\varepsilon \frac{dz}{dx} = \hat{B}(x,\varepsilon)z$ via the formal change of variables $y = \hat{T}(x,\varepsilon)z$.

Assumption A1 : $A(x,\varepsilon)$ is analytic on $\mathcal{D} \times D(0,\varepsilon_0)$.

Summability

The differential equation $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$, where

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Assumption A2 :

$$\mathbf{a}(x,\varepsilon) = \mathcal{O}(x^{\mu+\gamma}), \quad \mathbf{b}(x,\varepsilon) = \mathcal{O}(x^{\mu}) \quad \text{and} \quad \mathbf{c}(x,\varepsilon) = \mathcal{O}(x^{\mu+2\gamma}).$$

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Result. $\hat{T}$ and $\hat{B}$ are 1-summable in the direction $\arg\varepsilon = 0$.

Step 1 : Preparatory simplifications

$$(1) \quad \varepsilon \frac{dy}{dx} = A(x,\varepsilon)y \quad \text{where } A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix},$$

$$\downarrow \qquad y = T(x)u$$

$$(2) \quad \varepsilon \frac{du}{dx} = B(x,\varepsilon)u \quad \text{where } B_0(x) = \begin{pmatrix} -x^{p-1} & 0 \\ 0 & x^{p-1} \end{pmatrix},$$

$$\downarrow \qquad u = \hat{\Phi}(x,\varepsilon)v$$

$$(3) \quad \varepsilon \frac{dv}{dx} = \hat{C}(x,\varepsilon)v \quad \text{where } \hat{C}(x,\varepsilon) = \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}.$$

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Proposition. $\hat{\Phi}$ is 1-summable in the direction $\arg \varepsilon = 0$.

Proof.

$$\hat{\Phi} = \begin{pmatrix} 1 & \hat{\phi}^- \\ \hat{\phi}^+ & 1 \end{pmatrix}$$

The formal series $\hat{\phi}^+$ is the unique formal solution of

$$\varepsilon\phi' = 2x^{p-1}\phi + \varepsilon P(x, \phi, \varepsilon).$$

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Assumption A2 $\implies P(x, \phi, \varepsilon) = \mathcal{O}(x^{p-1})$.

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For a fixed x_0, consider the set of holomorphic and bounded functions on $M_{x_0} \times S_0$, where

$$S_0 = \left\{\varepsilon \in \mathbb{C},\ 0 < |\varepsilon| < \varepsilon_0 \text{ and } |\arg \varepsilon| < \frac{\pi}{2} - \delta\right\}$$

and

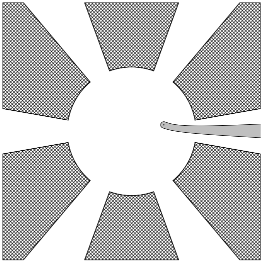

Figure: M_{x_0}

$\varepsilon\phi' = 2x^{p-1}\phi + \varepsilon P(x,\phi,\varepsilon)$

We apply the Banach fixed point theorem to the equation

$$\phi = \mathcal{T}\phi, \quad (\mathcal{T}\phi)(x,\varepsilon) = \frac{1}{\varepsilon}\int_\infty^x e^{\frac{2}{p\varepsilon}(x^p - t^p)} P(t,\phi(t,\varepsilon),\varepsilon)\,\mathrm{d}t.$$

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Its solution ϕ^+ admits a Gevrey-1 asymptotic expansion :

$$\phi^+(x, \varepsilon) \sim \hat{\phi}^+(x, \varepsilon) \quad \text{as } S_0 \ni \varepsilon \to 0, \text{ uniformly on } M_{x_0}.$$

$\varepsilon\phi' = 2x^{p-1}\phi + \varepsilon P(x, \phi, \varepsilon)$

Now proceed the same way for each $\xi \in M_{x_0}$.

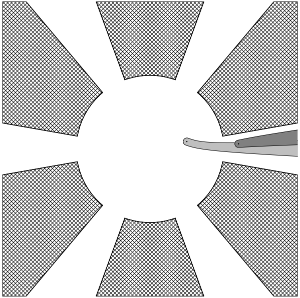

Figure: M_ξ in dark grey

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Combining the solutions of the analogous fixed point equation, we obtain an analytic function

$$\phi^+ : \; M_{x_0} \times S \; \to \mathbb{C},$$

where

$$S = \left\{ \varepsilon \in \mathbb{C}, \; 0 < |\varepsilon| < \varepsilon_0 \text{ and } |\arg \varepsilon| < \frac{\pi}{2} + \delta \right\}$$

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Therefore $\hat{\Phi}$ is 1-summable in the direction $\arg\varepsilon = 0$.

□

End of the proof

1. Preparation of equation (1)
2. Fundamental matrix solution
3. 1-summable slow fast factorization
4. Simplification