Uniform simplification in the full neighborhood of a turning point

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Plan of the talk

Introduction

Consider the differential equation

$$\varepsilon^2 \frac{d^2 y}{dx^2} - Q(x) y = 0,$$

where

- $\varepsilon > 0$, $\varepsilon \to 0$,
- $x \in [a, b]$,
- $Q : [a, b] \to \mathbb{R}$ of class C^1.

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Example

The Schrödinger equation (1925) :

$$\frac{d^2 y}{dx^2} - \frac{2m}{\hbar^2}(V(x) - E)y = 0.$$

Here $\hbar$ plays the role of ε and $Q(x) = 2m(V(x) - E)$.

Liouville-Green (1837)

$$\varepsilon^2 \frac{d^2 y}{dx^2} - Q(x)y = 0 \tag{1}$$

Approximation of solutions :
If $Q(x) > 0$,

$$\phi^{\pm}(x,\varepsilon) = Q(x)^{-\frac{1}{4}} \exp\left(\pm\frac{1}{\varepsilon}\int^{x}\sqrt{Q(\xi)}d\xi\right). \tag{2}$$

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If $Q(x) < 0$,

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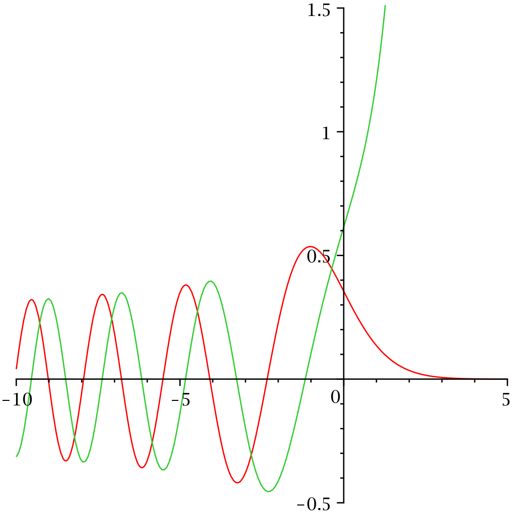
If $Q(x_0) = 0$ and $Q'(x_0) \neq 0$, then the functions (2) and (3) are no more approximations of the solutions.

Turning point

The zeros of $Q(x)$ separate regions with oscillating behavior from regions with exponential behavior.



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Definition

The zeros of $Q(x)$ are called *turning points.*

Mathematical background

Consider the differential equation

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y, \tag{4}$$

where

- x is a complex variable,
- ε is a small complex parameter,
- $A(x, \varepsilon)$ is a 2×2 matrix of holomorphic and bounded functions on $D(0, r_0) \times D(0, \varepsilon_0)$.

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Otherwise the point $x = 0$ is a turning point for system (4).

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Consider the differential system

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Let $A_0(x)$ be the matrix $A(x, 0)$.

We assume that :

- $A_0(0)$ admits a unique eigenvalue 0,
- $\operatorname{tr} A(x, \varepsilon) \equiv 0$,
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In this case $A_0(x)$ admits two distinct eigenvalues when $x \neq 0$, which are equal at $x = 0$.

Mathematical background

We can reduce the study to differential systems of this form

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y,$$

where

- $\operatorname{tr} A(x, \varepsilon) \equiv 0$,
- $A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+\nu} & 0 \end{pmatrix}$, with $\mu, \nu \in \mathbb{N}$ and $\mu\nu \neq 0$.

Condition ($\mathcal{C}$)

We consider the differential system

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y,$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} \mathbf{a}(x,\varepsilon) & \mathbf{b}(x,\varepsilon) \\ \mathbf{c}(x,\varepsilon) & -\mathbf{a}(x,\varepsilon) \end{pmatrix}.$$

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Condition $(\mathcal{C})$:

1. ν is even and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-2)})$,
2. ν is odd and $\mathbf{c}(x,0) = \mathcal{O}(x^{\frac{1}{2}(\nu-1)})$.

Simplification theorems

Hanson & Russell (1967)

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Theorem. If $(\mathcal{C})$ is satified, then there exists $\hat{T}(x,\varepsilon) = \sum_{n\geq 0} T_n(x)\varepsilon^n$, such that $\det T_0(x) \equiv 1$ and

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y \underset{y=\hat{T}(x,\varepsilon)z}{\sim} \varepsilon \frac{dz}{dx} = \hat{B}(x,\varepsilon)z,$$

where

$$\hat{B}(x,\varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} \hat{b}_{11}(x,\varepsilon) & \hat{b}_{12}(x,\varepsilon) \\ \hat{b}_{21}(x,\varepsilon) & \hat{b}_{22}(x,\varepsilon) \end{pmatrix}$$

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and the $\hat{b}_{ij}$ are polynomials in x :

$$\begin{aligned} \deg_x \hat{b}_{11} &< \mu, \\ \deg_x \hat{b}_{12} &< \mu, \\ \deg_x \hat{b}_{21} &< \mu+\nu, \\ \deg_x \hat{b}_{22} &< \mu. \end{aligned}$$

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- $\det T_0(x) \equiv 1$,
- $$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y \underset{y = T(x,\varepsilon) z}{\sim} \varepsilon \frac{dz}{dx} = B(x, \varepsilon) z$$

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  $$B(x, \varepsilon) = A_0(x) + \varepsilon \begin{pmatrix} b_{11}(x, \varepsilon) & b_{12}(x, \varepsilon) \\ b_{21}(x, \varepsilon) & -b_{11}(x, \varepsilon) \end{pmatrix}$$

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Recall :

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- Sibuya treated the case $A_0(x) = \begin{pmatrix} 0 & 1 \\ x^{\nu} & 0 \end{pmatrix}$, $\nu \in \mathbb{N}^{\star}$, in 1974.

Gevrey theory of composite asymptotic expansions

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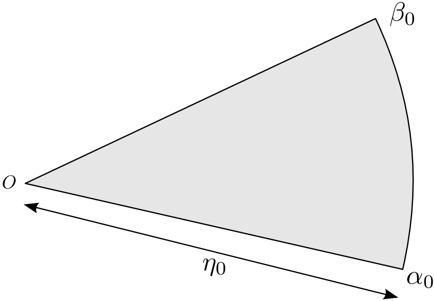
Let

- $S = \{\eta \in \mathbb{C},\ 0 < |\eta| < \eta_0 \text{ et } \alpha_0 < \arg \eta < \beta_0\}$,



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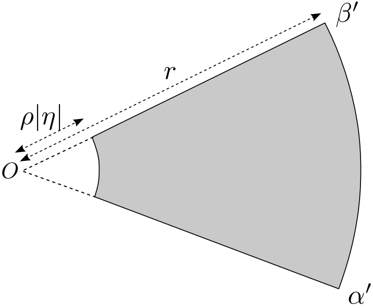

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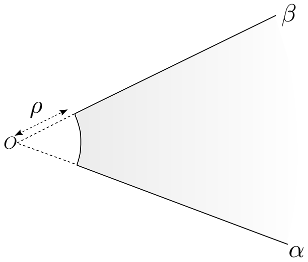

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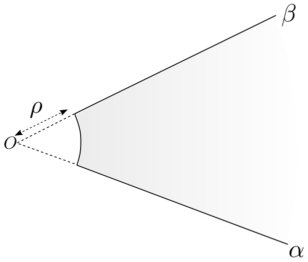

We call $(\mathcal{P})$ the following property :

$$\text{If } \eta \in S \text{ and } x \in V(\eta), \text{ then } \frac{x}{\eta} \in V.$$

Formal composite series

Definition

A *formal composite series* associated to V and $D(0,r)$ is a series of this form

$$\hat{y}(x,\eta) = \sum_{n\geq 0} \left(a_n(x) + g_n(\tfrac{x}{\eta}) \right) \eta^n,$$

where
the $a_n(x)$ are holomorphic and bounded functions on $D(0,r)$,
the $g_n(\mathbf{X})$ are holomorphic and bounded functions on V such that

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The series $\sum_{n\geq 0} a_n(x)\eta^n$ is called the *slow part* of $\hat{y}(x,\eta)$.
The series $\sum_{n\geq 0} g_n(\frac{x}{\eta})\eta^n$ is called the *fast part* of $\hat{y}(x,\eta)$.

Let $y(x,\eta)$ be a holomorphic and bounded function defined for $\eta \in S$ and for $x \in V(\eta)$, and let $\hat{y}(x,\eta) = \sum_{n\geq 0}\left(a_n(x) + g_n(\frac{x}{\eta})\right)\eta^n$ be a formal composite series.

Definition

We say that y admits $\hat{y}$ as composite asymptotic expansion (CAsE), as $\eta \to 0$ in S and $x \in V(\eta)$, if $\forall N \in \mathbb{N}$, $\exists K_N > 0$,

$$\left| y(x,\eta) - \sum_{n=0}^{N-1}\left(a_n(x) + g_n(\tfrac{x}{\eta})\right)\eta^n \right| \leq K_N |\eta|^N,$$

for all $\eta \in S$ and all $x \in V(\eta)$.

Gevrey CAsE

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We say that y admits $\hat{y}$ as CAsE of Gevrey order $\frac{1}{p}$, as $\eta \to 0$ in S and $x \in V(\eta)$, if $\exists C, L > 0$, $\forall N \in \mathbb{N}$,

$$\left| y(x,\eta) - \sum_{n=0}^{N-1} \left(a_n(x) + g_n(\tfrac{x}{\eta}) \right) \eta^n \right| \leq C L^N \Gamma(\tfrac{N}{p} + 1) |\eta|^N,$$

for all $\eta \in S$ and all $x \in V(\eta)$ and

$$g_n(\mathbf{X}) \sim_{\frac{1}{p}} \sum_{m>0} g_{nm} \mathbf{X}^{-m}, \text{ as } V \ni \mathbf{X} \to \infty$$

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Notation: $y(x,\eta) \sim_{\frac{1}{p}} \hat{y}(x,\eta)$, as $\eta \to 0$ in S and $x \in V(\eta)$.

Consistent good covering

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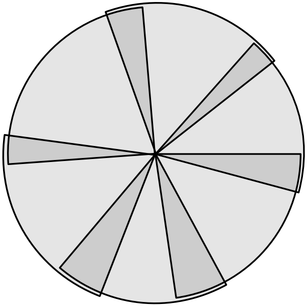

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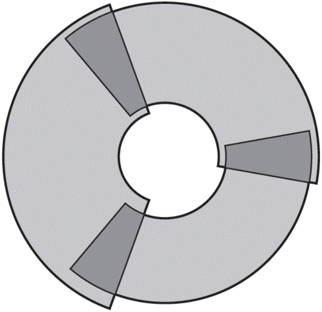

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- for all $\eta \in S_\ell$,
  $(V_\ell^j(\eta))_j$ is a consistent good covering of $\{x \in \mathbb{C},\ \rho|\eta| < |x| < r\}$,
- if $\eta \in S_\ell$ and $x \in V_\ell^j(\eta)$, then $\frac{x}{\eta} \in V^j$.

Let $S_\ell, V^j, V_\ell^j(\eta)$, $\ell = 1, \ldots, L$, $j = 1, \ldots, J$, be a consistent good covering and $V_\ell^j(\eta) \subset \tilde{V}_\ell^j(\eta)$. Let $\left(y_\ell^j(x,\eta)\right)_{j,\ell}$ be a collection of holomorphic and bounded functions defined for $\eta \in S_\ell$ and $x \in \tilde{V}_\ell^j(\eta)$ such that

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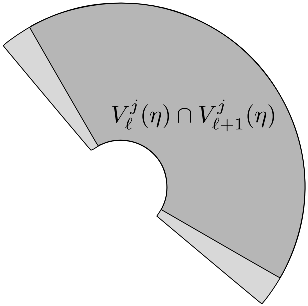
$$\left|\left(y_{\ell+1}^j - y_\ell^j\right)(x,\eta)\right| = \mathcal{O}\left(e^{-\frac{A}{|\eta|^p}}\right)$$

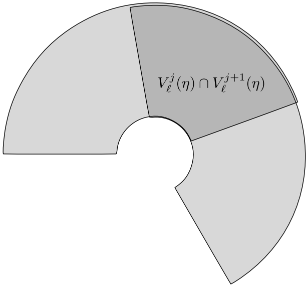


Theorem of Fruchard-Schäfke

A theorem of Ramis-Sibuya type

Let $S_\ell, V^j, V_\ell^j(\eta)$, $\ell = 1, \dots, L$, $j = 1, \dots, J$, be a consistent good covering and $V_\ell^j(\eta) \subset \tilde{V}_\ell^j(\eta)$. Let $\left(y_\ell^j(x,\eta)\right)_{j,\ell}$ be a collection of holomorphic and bounded functions defined for $\eta \in S_\ell$ and $x \in \tilde{V}_\ell^j(\eta)$ such that

$$\left|\left(y_\ell^{j+1} - y_\ell^j\right)(x,\eta)\right| = \mathcal{O}\left(\mathrm{e}^{-B\left|\frac{x}{\eta}\right|^p}\right)$$



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$$\left|\left(y^j_{\ell+1} - y^j_\ell\right)(x,\eta)\right| = \mathcal{O}\left(\mathrm{e}^{-\frac{A}{|\eta|^p}}\right)$$

and

$$\left|\left(y^{j+1}_\ell - y^j_\ell\right)(x,\eta)\right| = \mathcal{O}\left(\mathrm{e}^{-B\left|\frac{x}{\eta}\right|^p}\right).$$

Then

$$y^j_\ell(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0} \left(a_n(x) + g^j_n(\tfrac{x}{\eta})\right)\eta^n,$$

$$g^j_n(\mathbf{X}) \sim_{\frac{1}{p}} \sum_{m>0} g_{nm}\mathbf{X}^{-m}, \text{ as } V^j \ni \mathbf{X} \to \infty.$$

Proof of the main result

The case ν even

Assume that ν is even : $\nu = 2\gamma$.

Consider the differential system

$$\varepsilon \frac{dy}{dx} = A(x,\varepsilon) y, \tag{5}$$

where

$$A(x,\varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{a}(x,\varepsilon) & \mathbf{b}(x,\varepsilon) \\ \mathbf{c}(x,\varepsilon) & -\mathbf{a}(x,\varepsilon) \end{pmatrix}.$$

In this case, the condition $(\mathcal{C})$ becomes $\mathbf{c}(x,0) = \mathcal{O}(x^{\gamma-1})$.

Steps of the proof

1. Fundamental system of solutions
2. Slow-fast factorization of a CAsE
3. Analytic simplification

Fundamental system of solutions

Fundamental system of solutions

Proposition

Fundamental system of solutions of $\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y$ *:*

$$Y(x, \eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} Q(x, \eta) e^{\Lambda(x, \eta)},$$

where
η *is a root of* ε, $\varepsilon = \eta^p$, *with* $p = \mu + \gamma + 1$,
Q *admits a CAsE of Gevrey order* $\frac{1}{p}$, *as* $\eta \to 0$ *in* S *and* $x \in V(\eta)$,
Λ *is a diagonal matrix.*

Preparation

$$(1) \quad \varepsilon \frac{dy}{dx} = A(x,\varepsilon) y \quad \text{where } A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix},$$

$$\downarrow \qquad y = T(x) u$$

$$(2) \quad \varepsilon \frac{du}{dx} = B(x,\varepsilon) u \quad \text{where } B_0(x) = \begin{pmatrix} -x^{p-1} & 0 \\ 0 & x^{p-1} \end{pmatrix},$$

$$\downarrow \qquad u = \Phi(x,\eta) v \text{ and } \varepsilon = \eta^p$$

$$(3) \quad \eta^p \frac{dv}{dx} = C(x,\eta) v \quad \text{where } C(x,\eta) = \begin{pmatrix} -x^{p-1} + \dots & 0 \\ 0 & x^{p-1} + \dots \end{pmatrix}.$$

Existence of Φ

We precise now the second change of variables : $u = \Phi v$ and $\varepsilon = \eta^p$.

The matrix Φ is as follows :

$$\Phi = \begin{pmatrix} 1 & \phi^- \\ \phi^+ & 1 \end{pmatrix}.$$

Existence of Φ

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The matrix Φ is as follows :

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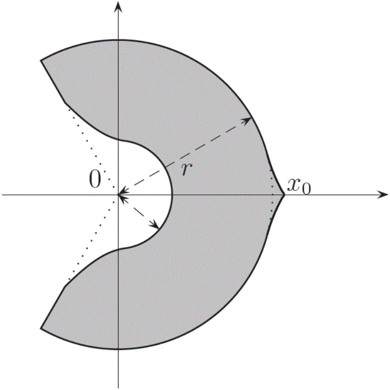
The function ϕ^+, resp. ϕ^-, satisfies a Riccati equation :

$$\eta^p \frac{d\phi}{dx} = \pm 2x^{p-1}\phi + F^{\pm}(\phi)(x,\eta).$$

Existence of ϕ^+

$$\eta^p \frac{d\phi^+}{dx} = 2x^{p-1}\phi^+ + F^+(\phi^+)$$

$\mathcal{M}_k = \{$holomorphic functions $\phi(x,\eta)$ defined for $\eta \in S$ and $x \in \Omega(\eta)$, $|\phi(x,\eta)| \leq k\}$



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Consider the following mapping $\mathcal{T} : \mathcal{M}_k \to \mathcal{M}_k$,

$$\phi \quad \mapsto \quad \frac{1}{\eta^p} \int_{\gamma_x} \mathrm{e}^{\frac{2}{p}\left(\frac{x^p}{\eta^p} - \frac{\xi^p}{\eta^p}\right)} F^+(\phi(\xi,\eta))\, \mathrm{d}\xi.$$

Existence of ϕ^+

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$$\begin{aligned} \text{Banach fixed-point theorem} \quad &\Rightarrow \quad \text{existence of } \phi^+ \\ &\Rightarrow \quad \text{existence of } (\phi^+)^j_\ell \end{aligned}$$

Existence of ϕ^+

$$\eta^p \frac{d\phi^+}{dx} = 2x^{p-1}\phi^+ + F^+(\phi^+)$$

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Banach fixed-point theorem $\quad\Rightarrow\quad$ existence of ϕ^+

$\quad\Rightarrow\quad$ existence of $(\phi^+)^j_\ell$

Theorem of Fruchard-Schäfke $\quad\Rightarrow\quad (\phi^+)^j_\ell(x,\eta) \sim_{\frac{1}{p}} (\hat{\phi}^+)^j(x,\eta)$

Summary

$$(1) \quad \varepsilon \frac{dy}{dx} = A(x,\varepsilon)y \quad \text{where } A_0(x) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu+2\gamma} & 0 \end{pmatrix},$$

$$\downarrow \qquad y = T(x)u$$

$$(2) \quad \varepsilon \frac{du}{dx} = B(x,\varepsilon)u \quad \text{where } B_0(x) = \begin{pmatrix} -x^{p-1} & 0 \\ 0 & x^{p-1} \end{pmatrix},$$

$$\downarrow \qquad u = \Phi(x,\eta)v \text{ and } \varepsilon = \eta^p$$

$$(3) \quad \eta^p \frac{dv}{dx} = C(x,\eta)v \quad \text{where } C(x,\eta) = \begin{pmatrix} -x^{p-1} + \dots & 0 \\ 0 & x^{p-1} + \dots \end{pmatrix}.$$

Fundamental system of solutions

We deduce the form of a fundamental system of solutions of $\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y$:

$$Y(x, \eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} Q(x, \eta) e^{\Lambda(x, \eta)},$$

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Q admits a CAsE of Gevrey order $\frac{1}{p}$, as $\eta \to 0$ in S and $x \in V(\eta)$,

$$\Lambda(x,\eta) = \begin{pmatrix} -\frac{1}{p}\frac{x^p}{\eta^p} + R_1(\varepsilon)\log x & 0 \\ 0 & \frac{1}{p}\frac{x^p}{\eta^p} + R_2(\varepsilon)\log x \end{pmatrix}.$$

Slow-fast factorization

Slow-fast factorization

Theorem

For all $r \in]0, r_0[$*, there exist* $L(x,\varepsilon)$ *holomorphic and bounded on* $D(0,r) \times \tilde{S}$ *and* $R(x,\eta)$ *holomorphic and bounded for* $\eta \in S$*,* $x \in V(\eta)$*, such that*

$$Q(x,\eta) = L(x,\varepsilon) \cdot R(x,\eta),$$

$$L(x,\varepsilon) \sim_1 \sum_{n\geq 0} A_n(x)\varepsilon^n, \text{ as } \varepsilon \to 0 \text{ in } \tilde{S} \text{ and } |x| < r,$$

and

$$R(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0} G_n(\tfrac{x}{\eta})\eta^n, \text{ as } \eta \to 0 \text{ in } S \text{ and } x \in V(\eta),$$

$$G_n(\mathbf{X}) \sim_{\frac{1}{p}} \sum_{m\geq 0} G_{nm}\mathbf{X}^{-m}, \text{ as } \mathbf{X} \to \infty \text{ in } V.$$

The matrix Y

The matrix Y

As $Q = L \cdot R$, we have

$$\begin{aligned} Y(x,\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} Q(x,\eta) e^{\Lambda(x,\varepsilon)}, \\ &= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} L(x,\varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & x^{-\gamma} \end{pmatrix}}_{P(x,\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & x^\gamma \end{pmatrix} R(x,\eta) e^{\Lambda(x,\varepsilon)}, \end{aligned}$$

Lemma

The matrix $Y(x,\eta)$ *can be written*

$$Y(x,\eta) = P(x,\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} R(x,\eta) e^{\Lambda(x,\varepsilon)},$$

where
P *is a slow matrix, i.e.*

$$P(x,\varepsilon) \sim_1 \sum_{n\geq 0} A_n(x)\varepsilon^n, \text{ as } \tilde{S} \ni \varepsilon \to 0, \ |x| < r,$$

R *is a fast matrix, i.e.*

$$R(x,\eta) \sim_{\frac{1}{p}} \sum_{n\geq 0} G_n\left(\tfrac{x}{\eta}\right)\eta^n, \text{ as } S \ni \eta \to 0, \ x \in V(\eta),$$

Λ *is a diagonal matrix.*

Analytic simplification

Proposition

The change of variables $y = P(x,\varepsilon)w$ *reduces the system* $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$ *to*

$$\varepsilon \frac{dw}{dx} = D(x,\varepsilon)w,$$

where $D(x,\varepsilon) \sim_1 \hat{D}(x,\varepsilon)$*,*

$$\hat{D}(x,\varepsilon) = \begin{pmatrix} \hat{d}_{11}(x,\varepsilon) & \hat{d}_{12}(x,\varepsilon) \\ \hat{d}_{21}(x,\varepsilon) & -\hat{d}_{11}(x,\varepsilon) \end{pmatrix},$$

and the $\hat{d}_{ij}$ *are polynomials in* x *such that*

$$\begin{aligned} \deg_x \hat{d}_{11} &\le \mu + \gamma, \\ \deg_x \hat{d}_{12} &= \mu, \\ \deg_x \hat{d}_{21} &= \mu + 2\gamma. \end{aligned}$$

Proof.

On the one hand,

$$D = P^{-1}AP - \varepsilon P^{-1}P'$$

and

$$D(x,\varepsilon) \sim_1 \hat{D}(x,\varepsilon),$$

as $\varepsilon \to 0$ in $\tilde{S}$ and $|x| < r$.

On the other hand, $W(x,\eta) = \begin{pmatrix} 1 & 0 \\ 0 & x^{\gamma} \end{pmatrix} R(x,\eta)e^{\Lambda(x,\eta)}$ is a fundamental system of solutions of equation $\varepsilon \frac{dw}{dx} = D(x,\varepsilon)w$ and

$$D(x,\varepsilon) = \varepsilon W'(x,\eta)W(x,\eta)^{-1}.$$

$\Rightarrow$ a bound for the degree of each entry of $\hat{D}(x,\varepsilon)$.

□

Let $\tilde{D} = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} \\ \tilde{d}_{22} & -\tilde{d}_{11} \end{pmatrix}$ be a matrix of polynomials in x such that

$$\tilde{D}(x,\varepsilon) \sim_1 \hat{D}(x,\varepsilon),$$

as $\varepsilon \to 0$ in $\tilde{S}$ and $|x| < r$, and

$$\begin{aligned} \deg_x \tilde{d}_{11} &\leq \mu + \gamma, \\ \deg_x \tilde{d}_{12} &= \mu, \\ \deg_x \tilde{d}_{21} &= \mu + 2\gamma. \end{aligned}$$

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Proposition

For all $r \in]0, r_0[$, there exists $\tilde{P}(x,\varepsilon)$, holomorphic and bounded on $D(0,r) \times \tilde{S}$, admitting an asymptotic expansion of Gevrey order 1, such that $\det P_0(x) \equiv 1$ and the change of variables $y = \tilde{P}(x,\varepsilon)w$ reduces the differential system $\varepsilon \frac{dy}{dx} = A(x,\varepsilon)y$ to

$$\varepsilon \frac{dw}{dx} = \tilde{D}(x,\varepsilon)w.$$

The main result (even case)

Theorem

If $(\mathcal{C})$ *is satisfied, then,* $\forall r \in]0, r_0[$ *and for all sufficiently small open sector* S *with vertex in 0, there exists a* 2×2 *holomorphic and bounded matrix* $T(x, \varepsilon)$ *on* $D(0, r) \times S$ *such that* $T(x, \varepsilon) \sim_1 \hat{T}(x, \varepsilon)$ *as* $\varepsilon \to 0$ *in* S *and* $|x| < r$, $\det T_0(x) \equiv 1$ *and*

$$\varepsilon \frac{dy}{dx} = A(x, \varepsilon) y \underset{y = T(x,\varepsilon) z}{\sim} \varepsilon \frac{dz}{dx} = B(x, \varepsilon) z$$

where

$$B(x, \varepsilon) = \begin{pmatrix} 0 & x^{\mu} \\ x^{\mu + 2\gamma} & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} b_{11}(x, \varepsilon) & b_{12}(x, \varepsilon) \\ b_{21}(x, \varepsilon) & -b_{11}(x, \varepsilon) \end{pmatrix},$$

and the b_{ij} *are polynomials in* x *such that*

$$\deg_x b_{11} < \mu, \; \deg_x b_{12} < \mu \text{ and } \deg_x b_{21} < \mu + 2\gamma.$$